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Entry flow in a channel. Part 2

By S. D. R. WILSON
Department of Mathematics, University of Manchester

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This paper complements an earlier paper by Van Dyke which has appeared under the same title. The problem of channel entry flow is re-examined and the early work is found to be formally incorrect. The techniques of modern boundary-layer theory are used to examine the region near the entrance. Various inlet conditions are considered and it is found that the usual condition of uniform entry velocity causes the intrusion of fractional powers of the Reynolds number into the expansions. The most satisfactory model is that of uniform flow into an infinite cascade of parallel plates.

The non-uniformity of the expansions at large downstream distances was studied in Van Dyke's paper and is not dealt with here, except to show that it may be treated separately.

1. Introduction

This paper is intended to be read together with a paper by Van Dyke (1970) which has already appeared under the same title; we here report complementary aspects of an independent investigation of the same problem. The problem is that of laminar, incompressible entry flow in a plane channel, and the object of both investigations was to review critically the early work of Schlichting and others in the light of modern boundary-layer theory.

In Van Dyke's paper, hereafter referred to as 'E 1', the historical background is described, and the essential physics of the situation elucidated. The argument is presented largely in physical terms. The main objectives of the present paper are to present the criticism of the early work from a more mathematical viewpoint, and to analyse the boundary-layer structure in greater detail for the various inlet conditions that have been considered.

The way in which the boundary layers eventually merge to form the ultimate parabolic velocity distribution was also considered in E 1, and the important conclusion was drawn, that it is in this region, rather than the entry, that Schlichting's analysis is relevant. This will not be studied here, except to show that it constitutes a (mathematically) separate question.

2. Formulation

The notation is that of E1. Lengths are referred to the half-width a of the channel, and velocities to the free-stream speed U. The Reynolds number R is Ua/ν. Then the dimensionless stream function ψ satisfies

$$\left(\psi_y \frac{\partial}{\partial x} - \psi_x \frac{\partial}{\partial y}\right) \nabla^2\psi = R^{-1}\nabla^4\psi. \tag{2.1}$$

The channel walls are at $y = \pm 1$ for $x \geqslant 0$ and the boundary conditions there are

$$\psi(x, \pm 1) = \pm 1, \quad \psi_y(x, \pm 1) = 0. \tag{2.2}$$

Various inlet conditions will be considered. On the assumption of uniform velocity at the channel entrance we have

$$\psi = y, \quad \psi_x = 0 \quad \text{at} \quad x = 0. \tag{2.3}$$

These boundary conditions, although somewhat artificial, have been widely used, in particular by Schlichting (1960), and by Atkinson & Goldstein in their work on the circular pipe (Goldstein 1965, p. 305). As noted in E1, this introduces weak vorticity, of order $R^{-\frac{1}{2}}$, into the inviscid core and is responsible for the intrusion of various fractional powers of R into the expansions.

Some improvement can be gained by assuming that the vorticity is zero at $x = 0$:

$$\psi = y, \quad \psi_{xx} = 0 \quad \text{at} \quad x = 0. \tag{2.4}$$

Now the inviscid flow is irrotational but (as we shall see) there is a singularity in the inviscid speed on the line $x = 0$, due to the suppression of upstream influence of the plates.

No doubt the most realistic tractable model is that of an infinite cascade of parallel plates, as described in E1, §2. No conditions are imposed at $x = 0$ and instead we have

$$\psi(x, \pm 1) = \pm 1, \quad \psi_{yy}(x, \pm 1) = 0 \quad \text{for} \quad x < 0 \tag{2.5}$$

and

$$\psi \sim y \quad \text{as} \quad x \to -\infty.$$

Again the inviscid flow is irrotational but now it is also free of singularities.

The boundary-value problems arising from these different ideas will be solved together in the next section. The aim is to construct asymptotic expansions for large R, and four distinct regions (see figure 1) are found as shown in E1. We shall be concerned with regions I and II, the inviscid core and boundary layer.

This investigation can proceed independently of the other two regions. The leading edge region O is analytically intractable, and the uncertainty about the details of the flow there is reflected in the appearance in the boundary-layer solution of undetermined constants multiplying eigensolutions. However, the structure of the boundary-layer expansion can be found readily enough, and it will turn out that the formal difficulties prevent the solution being carried to the point where the first of the constants makes its appearance.

More important is the non-uniformity of the boundary-layer expansion as $x \to \infty$ (region III). The boundary layers meet when $x = O(R)$ and writing $\xi = x/R$ yields equation (2.6) of E 1:

$$\left(\psi_y \frac{\partial}{\partial \xi} - \psi_\xi \frac{\partial}{\partial y}\right) \psi_{yy} = \psi_{yyyy} + O(R^{-2}). \tag{2.6}$$

The 'initial' conditions, at $\xi = 0$, are found by letting $x \to \infty$ in the full inviscid solution, and it will not be necessary to match backwards (Van Dyke 1964, p. 94). On the other hand the inviscid equations are elliptic, but the missing

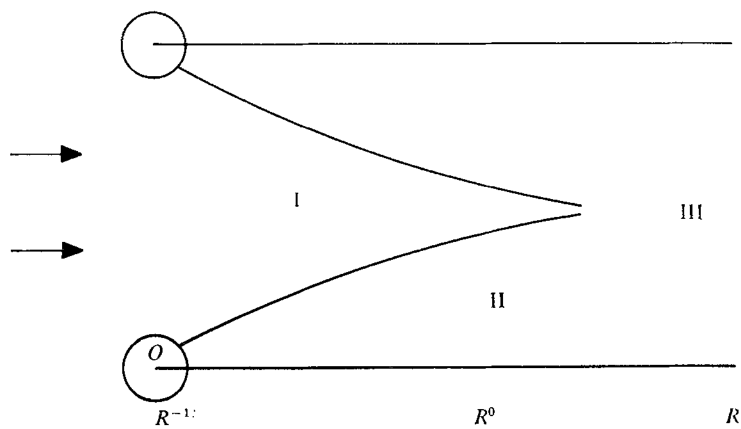

FIGURE 1. Sketch of the co-ordinate system and the asymptotically distinct regions in the channel.

boundary condition (at $x = \infty$) is essentially that the solution be matchable, which is enough to rule out exponentially large solutions. This transition is considered in detail in E 1. The further transition, from region III to the fully developed flow, results in an eigenvalue problem first considered by Schlichting, and in more detail by Wilson (1969).

The problem to be solved here, then, consists of the inviscid equation

$$\left(\psi_y \frac{\partial}{\partial x} - \psi_x \frac{\partial}{\partial y}\right) \nabla^2 \psi = O(R^{-1}), \tag{2.7}$$

with the condition that the normal velocity vanish on the walls, and the boundary-layer equation (for the wall $y = -1$)

$$\left(\Psi_\zeta \frac{\partial}{\partial x} - \Psi_x \frac{\partial}{\partial \zeta}\right) \Psi_{\zeta\zeta} = \Psi_{\zeta\zeta\zeta\zeta} + O(R^{-1}), \tag{2.8}$$

where $\Psi = R^{\frac{1}{2}}(\psi + 1)$ and $\zeta = R^{\frac{1}{2}}(y+1)$, with the conditions $\Psi = \Psi_\zeta = 0$ on $\zeta = 0$.

It is convenient at this point to return to the discussion of the early work on this problem. Schlichting's solution proceeds as follows. The expansion for the boundary layer has the form

$$\Psi = R^{\frac{1}{2}} \sum_{n=1}^{\infty} \left(\frac{x}{R}\right)^{\frac{1}{2}n} f_n(\eta), \tag{2.9}$$

where η is the usual similarity variable $\zeta/(2x)^{\frac{1}{2}}$ and the f_n are functions to be determined. This series is substituted into the boundary-layer equation (2.8). The fluid in the inviscid core is accelerated and this is allowed for by means of the expansion

$$\psi = y\left\{1+\sum_{n=1}^{\infty} K_n(x/R)^{\frac{1}{2}n}\right\}, \tag{2.10}$$

where the K_n are constants found essentially by matching with (2.9). However it is apparent that the terms of (2.10) do not satisfy the appropriate equation (2.7) (except for the first). The essential point is that (2.10) is intended to be an expansion for $R \to \infty$ with x fixed, *not* for small x with R fixed. It would be more in the spirit of Schlichting's work to regard these expansions as intended for region III for small ξ as suggested by the use of the variable x/R. The resulting boundary-value problem, namely (2.6) with (2.2), is indeed the one posed by Schlichting, as noted in E 1, but here the inviscid core has disappeared and (2.10) must be abandoned.

It is illuminating to examine these ideas a little further in their original context. The displacement thickness $\delta(x)$ may be defined by

$$\int_0^1 (U-u)\,dy = U\delta,$$

where u is the streamwise velocity and U is the average velocity in the inviscid core. Using the continuity equation and noting that $U(0) = 1$, we obtain

$$U(x) = 1+\delta+\delta^2+\ldots.$$

The next step is to assume that δ is proportional to $(x/R)^{\frac{1}{2}}$ by analogy with the Blasius solution, and this suggests the form of (2.10).

This result, although correct, is inappropriate; the boundary-layer solution must match not into the average inviscid speed but into the speed at the edge of the boundary layer, which is not the same. The variations in second-order inviscid speed cannot be ignored when matching with boundary-layer terms of the same order.

3. The boundary-layer calculation

Whichever of the inlet conditions is adopted, the first-order inviscid solution is simply $\psi = y$, and hence the first-order boundary-layer solution will be the usual Blasius solution. Writing $\Psi = (2x)^{\frac{1}{2}} f_1(\eta)$ with $\eta = \zeta/(2x)^{\frac{1}{2}}$ gives

$$f_1^{\text{iv}}+f_1'f_1''+f_1f_1'''=0 \tag{3.1}$$

in the usual way. As $\eta \to \infty$ we have $f_1 \sim \eta-\beta+$ exponentially small terms, where $\beta = 1{\cdot}21678$. The flow due to the displacement thickness is represented by the second term in the inviscid expansion,

$$\psi = y+R^{-\frac{1}{2}}\psi_2+\ldots,$$

where ψ_2 satisfies

$$\left.\begin{aligned}\frac{\partial}{\partial x}\nabla^2\psi_2 &= 0\\ \psi_2 &= \pm\beta x^{\frac{1}{2}} \quad \text{on} \quad y=\pm 1, \quad x \geqslant 0.\end{aligned}\right\} \tag{3.2}$$

and

For the infinite cascade model the equation can be integrated and the arbitrary function of y identified (as zero) by letting $x \to -\infty$. In the other two cases this is not possible and instead there are two conditions at $x = 0$. We now deal with the three cases separately.

(i) *Infinite cascade*

The boundary conditions are (2.5). This problem is most conveniently solved by means of generalized functions and tables of the relevant Fourier transforms are given in Gel'fand & Shilov (1964). The solution is

$$\psi_2 = \frac{\beta}{(8\pi)^{\frac{1}{2}}}\int_0^\infty \frac{\sin\alpha x - \cos\alpha x}{\alpha^{\frac{3}{2}}}\frac{\sinh\alpha y}{\sinh\alpha}\,d\alpha. \tag{3.3}$$

To calculate the second-order boundary layer it is necessary to estimate ψ_2 when y is near -1. We find

$$\psi_2 = -\beta x^{\frac{1}{2}} + \beta(8\pi)^{\frac{1}{2}}(y+1)\int_0^\infty \frac{\sin\alpha x - \cos\alpha x}{\alpha^{\frac{1}{2}}}(\coth\alpha - 1)\,d\alpha, \tag{3.4}$$

where the integral $$\int_0^\infty \alpha^{-\frac{1}{2}}(\sin\alpha x - \cos\alpha x)\,dx = 0$$

has been subtracted to give an absolutely convergent integral.

The second term in (3.4) gives the inviscid speed to which the second-order boundary layer must match. The absence of singularities in it indicates that the expansion will proceed in the usual way. Rewriting (2.8) with x and η as independent variables, inserting the expansion

$$\Psi = (2x)^{\frac{1}{2}}f_1(\eta) + R^{-\frac{1}{2}}g(x,\eta) + \ldots$$

and retaining terms up to order $R^{-\frac{1}{2}}$ gives

$$g_{\eta\eta\eta\eta} + f_1 g_{\eta\eta\eta} + 2f_1' g_{\eta\eta} + f_1'' g_\eta + 2x(f_1''' g_x - f_1' g_{\eta\eta x}) = 0. \tag{3.5}$$

The boundary conditions are $g = g_\eta = 0$ at $\eta = 0$ and $g \sim \eta F(x)$ as $\eta \to \infty$, where $F(x)$ is the inviscid speed multiplied by $(2x)^{\frac{1}{2}}$:

$$F(x) = \tfrac{1}{2}\beta x^{\frac{1}{2}}\pi^{-\frac{1}{2}}\int_0^\infty \frac{\sin\alpha x - \cos\alpha x}{\alpha^{\frac{1}{2}}}(\coth\alpha - 1)\,d\alpha. \tag{3.6}$$

It is possible to obtain a solution for small x by expanding g and F in power series:

$$F(x) = \tfrac{1}{2}\beta\pi^{-\frac{1}{2}}x^{\frac{1}{2}}\sum_0^\infty a_n x^n, \tag{3.7}$$

$$g(x,\eta) = x^{\frac{1}{2}}\sum_0^\infty x^n g_n(\eta). \tag{3.8}$$

The coefficients a_n are obtained in terms of an integral (which will be referred to again)

$$I_\nu = \int_0^\infty t^\nu(\coth t - 1)\,dt$$

$$= 2^{-\nu}\Gamma(\nu+1)\,\zeta(\nu+1),$$

where ζ is the Riemann ζ-function. The integral for a_0 diverges but may be obtained in the generalized sense. We have

$$a_n = \frac{\epsilon_n}{n!} I_{n-\frac{1}{2}}, \tag{3.9}$$

where $\epsilon_n = 1$ if $n = 1, 2, 5, 6, \ldots$, and -1 if $n = 0, 3, 4, 7, 8, \ldots$.

The functions g_n satisfy the equation

$$\left\{\frac{d^4}{d\eta^4} + f_1 \frac{d^3}{d\eta^3} + (1-2n) f_1' \frac{d^2}{d\eta^2} + f_1' \frac{d}{d\eta} + (1+2n) f_1'''\right\} g_n = 0. \tag{3.10}$$

It may be shown that four independent solutions of this equation can be found having, as $\eta \to \infty$, the asymptotic forms 1, η, η^{2n+1}, exponentially small. An exception is the case $n = 0$, when the third solution has the form $\eta \log \eta$. The boundary condition at infinity is

$$g_n'(\infty) = \tfrac{1}{2}\beta\pi^{-\frac{1}{2}} a_n \tag{3.11}$$

and this serves to rule out the third solution in each case (including $n = 0$). Some numerical integrations have been carried out and the results will be discussed at the end of the next section.

It might be expected that the power series (3.7) has only a finite radius of convergence; and it may be shown that as $n \to \infty$, $a_n = O(2^{-n})$, which indicates that this radius is 2. A representation of the solution for general x, from which the behaviour as $x \to \infty$ may be deduced, will now be given. A useful preliminary is to obtain the asymptotic expansion of $F(x)$ as $x \to \infty$. This is

$$F(x) \sim -\beta x + O(x^{-1}). \tag{3.12}$$

The boundary-layer equation (3.5) may be integrated once to give

$$g_{\eta\eta\eta} + f_1 g_{\eta\eta} + f_1' g_\eta + 2x(f_1'' g_x - f_1' g_{\eta x}) = F - 2xF'. \tag{3.13}$$

The Mellin transform $\tilde{g}(s, \eta)$, defined by

$$\tilde{g}(s, \eta) = \int_0^\infty g(x, \eta)\, x^{s-1}\, dx$$

must satisfy

$$L_s\{\tilde{g}\} = (2s+1)\tilde{F}(s), \tag{3.14}$$

where L_s is the operator

$$\frac{d^3}{d\eta^3} + f_1 \frac{d^2}{d\eta^2} + (2s+1) f_1' \frac{d}{d\eta} - 2s f_1'',$$

and $\tilde{F}(s)$ is the Mellin transform of $F(x)$.

An examination of the series (3.7) and (3.12) shows that the Mellin transform of $F(x)$ does not converge, but exists only in the generalized sense. In fact the right-hand side of (3.13) has a Mellin transform, and $\tilde{F}(s)$ may be defined by analytic continuation as indicated in (3.14). The main difficulty is in the choice of contour for the inverse transform

$$F(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{F}(s)\, x^{-s}\, ds,$$

for which one would normally choose c to lie in the region of convergence of the transform.

It is possible to avoid this difficulty and obtain results in terms of ordinary functions by subtracting from $F(x)$ the leading term in its power series, making use of the linearity of (3.5). The resulting function, $F_1(x)$ say, has an integral representation analogous to (3.6):

$$F_1(x) = \tfrac{1}{2}\beta\pi^{-\frac{1}{2}}x^{\frac{1}{2}}\int_0^\infty \frac{\sin\alpha x - \cos\alpha x + 1}{\alpha^{\frac{1}{2}}}(\coth\alpha - 1)\,d\alpha. \tag{3.15}$$

This has a Mellin transform

$$\tilde{F}_1(s) = \beta\pi^{-\frac{1}{2}}2^{s+\frac{1}{2}}\sin\tfrac{1}{2}\pi s\,\Gamma(s+\tfrac{1}{2})\,\Gamma(-s)\,\zeta(-s), \tag{3.16}$$

which converges in the strip $-\frac{3}{2} < \mathrm{Re}\,(s) < -1$.

Now let $G_s(\eta)$ be the solution of

$$L_s(\tilde{g}) = 2s+1 \tag{3.17}$$

which satisfies $G_s(0) = G_s'(0) = 0$ and $G_s'(\infty) = 1$. This is well defined except at eigenvalues. Then

$$\tilde{g}(s,\eta) = G_s(\eta)\,\tilde{F}_1(s)$$

and inverting the transform, we have

$$g(x,\eta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\tilde{F}_1(s)\,G_s(\eta)\,x^{-s}\,ds, \tag{3.18}$$

where $-\frac{3}{2} < c < -1$. The singularities of the integrand are the poles of $\tilde{F}_1$, which correspond to forced terms in the solution, and the poles of G_s, which correspond to eigensolutions. (We assume that G_s has no other singularities).

The poles $\tilde{F}_1(s)$ consist of two groups, at $s = -\frac{3}{2}, -\frac{5}{2}, -\frac{7}{2}, \ldots$, corresponding to the power series of $F_1(x)$ for small x, and at $s = -1, -\frac{1}{2}, 1, 3, 5, \ldots$ etc., corresponding to the asymptotic expansion of $F_1(x)$ for large x.

By completing the contour with a large left-hand semicircle the power series solution (3.7) and (3.8) is recovered. By shifting the contour to the right an asymptotic solution for large x is obtained; in this case the poles of G_s must be accounted for. These are all real and positive (Libby & Fox 1963); the first few are at $s = \frac{1}{2}$, 1·387, 2·314, 3·258. To determine the multiple of each eigensolution it is necessary to evaluate the residues of $G_s(\eta)$ at its poles; this involves a study of the equation adjoint to (3.14) and would in general require substantial numerical work. However, the first eigensolution, for which $s = \frac{1}{2}$, is known, (it is $\eta f_1' - f_1$), and an analytic solution is possible.

$G_s(\eta)$ satisfies (3.17); to examine the solution near $s = \frac{1}{2}$, we put $s = \frac{1}{2} + \epsilon$ and

$$G_s(\eta) = \epsilon^{-1}k(\eta f_1' - f_1) + h_0(\eta) + \epsilon h_1(\eta) + \ldots, \tag{3.19}$$

where k is a constant, to be determined. The coefficient of ϵ^{-1} satisfies the homogeneous equation $L_{\frac{1}{2}}(G_{\frac{1}{2}}) = 0$ and homogeneous boundary conditions. Next, the function h_0 satisfies

$$L_{\frac{1}{2}}(h_0) = 2(1 + kf_1'''), \tag{3.20}$$

with $h_0(0) = h_0'(0) = 0$, $h'(\infty) = 1$.

As $\eta \to \infty$, $f_1 \sim \theta + A\theta^{-2}\exp(-\tfrac{1}{2}\theta^2)$, where $A = 0{\cdot}331$, $\theta = \eta - \beta$, and the asymptotic form of (3.20) is

$$h_0''' + \theta h_0'' + 2h_0' = 2, \tag{3.21}$$

where the neglected terms are assumed to be exponentially small. Thus

$$h_0' + \theta h_0 = \theta^2 + B\theta + C$$

and so
$$h_0 \sim \theta + B + (C-1)(\theta^{-1} + \theta^{-3} + \ldots). \tag{3.22}$$

To define h_0 properly we must, therefore, demand $h_0' \to 1$ exponentially, so that $C = 1$. Now

$$\begin{aligned} C &= \lim_{\theta\to\infty}\{h_0' + \theta h_0 - \theta^2 - B\theta\} \\ &= \lim\{\theta^2 - \theta h_0'' - (\theta^2 - 1)h_0'\} \\ &= \lim\{f_1^2 - f_1 h_0'' - (f_1^2 - 1)h_0'\} \\ &= \int_0^\infty \{2f_1 f_1' - f_1 h_0''' - f_1' h_0'' - (f_1^2 - 1)h_0'' - 2f_1 f_1' h_0'\}\,d\eta. \end{aligned}$$

Using (3.20) we find
$$\begin{aligned} C &= 2\int_0^\infty f_1(f_1' - 1 - kf_1''')\,d\eta \\ &= 2\int_0^\infty \left\{f_1 f_1' + \frac{f_1'''}{f_1''} - kf_1' f_1'''\right\} d\eta \\ &= 2\log(A/\alpha) + k, \end{aligned}$$

where $\alpha = f_1''(0)$. Hence $k = 1{\cdot}70$. The contribution from the pole at $s = \tfrac{1}{2}$ is therefore

$$k\tilde{F}(\tfrac{1}{2})(\eta f_1' - f_1) = -2^{\frac{3}{2}}\beta k\zeta(-\tfrac{1}{2})\,x^{-\frac{1}{2}}(\eta f' - f). \tag{3.23}$$

This may be compared with the asymptotic expansion of the forced solution. An extraneous term of order $x^{\frac{1}{2}}$ has been introduced by the switch from $F(x)$ to $F_1(x)$ described above, and if this is corrected for, the first two terms are

$$\beta(2x)^{\frac{1}{2}}\{G_{-1}(\eta)\,x^{\frac{1}{2}} + \tfrac{1}{12}x^{-\frac{3}{2}}G_1(\eta) + \ldots\}. \tag{3.24}$$

Thus we have shown that the dominant term at infinity is the forced term of order x, followed by an eigensolution of order $x^{-\frac{1}{2}}$, (3.23), and then by the $O(x^{-1})$ forced term in (3.24). Then comes the eigensolution of order $x^{-1{\cdot}387}$; the numerical coefficient has not been determined here. (The procedure described above for the calculation of the constant k appears not to work for higher eigenvalues, and a numerical calculation would be necessary.)

(ii) *Irrotational entry*

The boundary conditions are (2.4) and the solution for ψ_2 is

$$\psi_2 = \frac{\beta}{(2\pi)^{\frac{1}{2}}}\int_0^\infty \frac{\sin\alpha x}{\alpha^{\frac{3}{2}}}\,\frac{\sinh\alpha y}{\sinh\alpha}\,d\alpha. \tag{3.25}$$

The analysis proceeds on much the same lines as the cascade problem and we shall give only an outline of the distinguishing features, which are concerned with the flow near $x = 0$.

The integral in (3.25) may be estimated near $y = -1$ in a similar manner to (3.3), this time using the result

$$\int_0^\infty \alpha^{-\frac{1}{2}} \sin \alpha x \, d\alpha = (\pi/2x)^{\frac{1}{2}}.$$

We find

$$\psi_2 = -\beta x^{\frac{1}{2}} + \frac{\beta}{(2\pi)^{\frac{1}{2}}}(y+1)\int_0^\infty \frac{\sin \alpha x}{\alpha^{\frac{1}{2}}} \coth \alpha \, d\alpha + \ldots$$

$$= -\beta x^{\frac{1}{2}} + \beta(y+1)\left\{\frac{1}{2x^{\frac{1}{2}}} + \sum_1^\infty b_n x^n\right\} + \ldots, \tag{3.26}$$

where $b_n = 0$ if n is even, and if n is odd,

$$b_n = \frac{(-1)^{\frac{1}{2}(n-1)}}{n!} I_{n-\frac{1}{2}}.$$

n	$g_n''(0)$	A_n
$-\frac{3}{2}$	$-0{\cdot}605$	$2{\cdot}404$
$-\frac{1}{2}$	0	$0{\cdot}441$
0	$0{\cdot}498$	$-0{\cdot}860$
$\frac{1}{2}$	$1{\cdot}172$	$1{\cdot}120$
1	$2{\cdot}041$	$2{\cdot}755$
2	$3{\cdot}178$	$4{\cdot}788$
3	$4{\cdot}145$	$6{\cdot}264$
4	$5{\cdot}012$	$7{\cdot}448$
5	$5{\cdot}811$	$8{\cdot}449$

TABLE 1. Results of numerical integration of (3.10). As $\eta \to \infty$, $g_n \sim \eta + A_n$. The entries for $n = -\frac{3}{2}$ and $n = \frac{1}{2}$ are needed for (3.24), the remainder for (3.8) and (3.27)

Again the integral in (3.25) is uniformly convergent and there is no singularity in the inviscid speed $(\partial\psi_2/\partial y)$, except at $x = 0$, a fact which reflects the artificial imposition of a boundary condition there. The second-order boundary layer may again be obtained as an infinite series

$$R^{-\frac{1}{2}}\Psi = \psi + 1 = R^{-\frac{1}{2}}(2x)^{\frac{1}{2}} f_1(\eta) + R^{-1}x^{\frac{1}{2}}\left\{x^{-\frac{1}{2}}h_{-\frac{1}{2}}(\eta) + \sum_1^\infty x^n h_n(\eta)\right\}. \tag{3.27}$$

The functions h_ν ($\nu = -\frac{1}{2}, 1, 2, 3, \ldots$) satisfy the same equation, (3.10), with n replaced by ν. Again the four independent solutions have the asymptotic forms 1, η, $\eta^{2\nu+1}$, exponentially small, except when $\nu = -\frac{1}{2}$, in which case the third solution behaves like $\log \eta$. The boundary conditions at infinity are

$$h'_{-\frac{1}{2}}(\infty) = \beta/2^{\frac{1}{2}},$$
$$h'_n(\infty) = 2^{\frac{1}{2}}\beta,$$

and while this rules out the third solution in general there is no immediate reason to suppose it will in the case $\nu = -\frac{1}{2}$. However, it is not difficult to see that if the logarithmic solutions were present it would be required, for the third-order inviscid flow, to find a potential function which behaves like $\log y$ on the x axis. Such a function, if it exists, would be physically unrealistic, highly oscillatory and presumably 'unmatchable' (cf. Goldstein 1960, p. 139). Accordingly this solution is required to be absent.

Note that when n is even, $b_n = 0$ and so $h_n \equiv 0$ because n is not an eigenvalue (Van Dyke 1964, p. 132).

It is clear that the h_n differ from the corresponding g_n only by a constant of proportionality. For the purposes of numerical integration they may all be normalized so that $h_n \sim \eta + A_n +$ small terms and the first few constants are given in table 1. It is worth remarking that the numerical integration in the case $\nu = -\frac{1}{2}$ would not be particularly straightforward were it not possible to integrate the equation twice immediately in this case.

(iii) *Uniform entry*

Using the conditions (2.3), we find

$$\psi_2 = \frac{\beta}{(2\pi)^{\frac{1}{2}}} \int_0^\infty \frac{1-\cos\alpha x}{\alpha^{\frac{3}{2}}} \frac{\sinh\alpha y}{\sinh\alpha} d\alpha. \tag{3.28}$$

As in the previous case the inviscid speed is singular at $x = 0$ but this is now less important than the singularity at $y = -1$ (cf. E1, §4). The integral for $\partial\psi_2/\partial y$ is not uniformly convergent and some care is needed to estimate it. The result is, near $y = -1$

$$\psi_2 = -\beta x^{\frac{1}{2}} + \beta\{2(y+1)\}^{\frac{1}{2}} + O(y+1), \tag{3.29}$$

and, as noted in E1, requires that the next term in the boundary-layer expansion shall be of relative order $R^{-\frac{1}{4}}$, and that the solution there must behave like $\eta^{\frac{1}{2}}$ at infinity. The thickness of the second-order boundary layer is proportional to $x^{\frac{1}{4}}$. If the detailed calculations are carried out up to this point it is possible to spot the general term of both the boundary layer and inviscid expansions and we confine ourselves to presenting the result. To avoid clumsy superscripts write $\tau_n = 2^{-n}$, $\sigma_n = 1-\tau_n$. The inviscid expansion is

$$\psi = y + R^{-\frac{1}{2}}\psi_2 + R^{-\frac{3}{4}}\psi_3 + \ldots + R^{-\sigma_n}\psi_{n+1} + \ldots \tag{3.30}$$

and the boundary-layer expansion is

$$R^{-\frac{1}{2}}\Psi = \psi + 1 = R^{-\frac{1}{2}}(2x)^{\frac{1}{2}} f_1(\eta) + R^{-\frac{3}{4}} x^{\frac{1}{4}} f_2(\eta) + \ldots + R^{-\sigma_n} x^{\tau_n} f_n(\eta) + \ldots. \tag{3.31}$$

Each f_n, except f_1, satisfies a homogeneous equation, which is (3.10) with the coefficients $\frac{1}{2}-n$ and $\frac{1}{2}+n$ replaced by σ_n and τ_n respectively. The four independent solutions have the asymptotic forms 1, η, $\eta^{2\tau_n}$, exponentially small; and matching rules out the *second* solution in this case. At infinity, then, we have

$$f_n \sim A_n \eta^{2\tau_n} + B_n + \ldots.$$

The solution for the general term of (3.30) is

$$\psi_{n+1} = \frac{\tau_n B_n}{\Gamma(\sigma_n)\cos(\frac{1}{2}\tau_n\pi)} \int_0^\infty \frac{1-\cos\alpha x}{\alpha^{\tau_n+1}} \frac{\sinh\alpha y}{\sinh\alpha} d\alpha. \tag{3.32}$$

The constants B_n are found by numerical integration of (3.10) with the new boundary conditions, and the results are given in table 2. This actually gives B_n/A_n; the constants A_n are found from the matching process, which gives

$$A_{n+1} = -B_n \sec(\tfrac{1}{2}\tau_n\pi).$$

An infinite sequence of powers of R is inserted between $R^{-\frac{1}{2}}$ and R^{-1}; a pathological situation, no doubt, but there is some rarity value in an asymptotic expansion of which the general term can be written down!

Finally, we note the skin-friction coefficients in the three cases. Defining $c_f = \tau/\frac{1}{2}\rho U^2$ in the usual way we find:

(i) the cascade model,

$$c_f = \frac{1}{(2Rx)^{\frac{1}{2}}} f_1''(0) + \frac{1}{Rx^{\frac{1}{2}}} \sum_0^\infty x^n g_n''(0) + \ldots;$$

(ii) the irrotational entry

$$c_f = \frac{1}{(2Rx)^{\frac{1}{2}}} f_1''(0) + \frac{1}{Rx} h_{-\frac{1}{2}}''(0) + \frac{1}{Rx^{\frac{1}{2}}} \sum_1^\infty x^n h_n''(0) + \ldots;$$

(iii) uniform entry

$$c_f = \frac{1}{(2Rx)^{\frac{1}{2}}} f_1''(0) + \frac{1}{(Rx)^{\frac{3}{4}}} f_2''(0) + \frac{1}{(Rx)^{\frac{7}{8}}} f_3''(0) + \ldots.$$

The values of the various derivatives may be found from tables 1 and 2.

n	α_n	$f_n''(0)$
2	−0·100	0·229
3	−0·525	0·116
4	−0·751	0·060
5	−0·871	0·031
6	−0·935	0·015
7	−0·967	0·008

TABLE 2. Results of numerical integration of (3.10) for uniform entry. As $\eta \to \infty, f_n \sim \eta^{2\tau_n} + \alpha_n + \ldots, (\tau_n = 2^{-n})$

Figure 2 shows the variation of the second-order correction to the skin-friction coefficient with x for the cascade model. For small x this is given by the series $x^{-\frac{1}{2}} \sum_0^\infty x^n g_n''(0)$ truncated at 3, 4 and 5 terms. For large x the asymptotic solution derived from (3.23) and (3.24) is used.

The corresponding series for the irrotational entry model will have similar accuracy if truncated at 5 terms. We may note that the apparently more singular term in the skin-friction coefficient $(Rx)^{-1} h_{-\frac{1}{2}}''(0)$ is actually absent because $h_{-\frac{1}{2}}''(0) = 0$.

4. Concluding remarks

The boundary-layer expansion has been calculated as far as terms of order R^{-1} for various inlet conditions. It appears that the most commonly studied model (uniform entry) is the least satisfactory from this point of view; and the most suitable model (the cascade) has received little attention until the present work and that of Van Dyke.

Attention has been mostly confined to the region in which x is $O(1)$. The early work of Schlichting and others has been shown to be incorrectly applied to this region, and is in fact a correct formulation of the problem in the region where x is $O(R)$.

The matching of the boundary layer and the inviscid flow is carried out in the case of uniform entry with only algebraically small error; this is unusual in problems of this type and is no doubt the result of the presence of vorticity induced in the core by the inlet conditions.

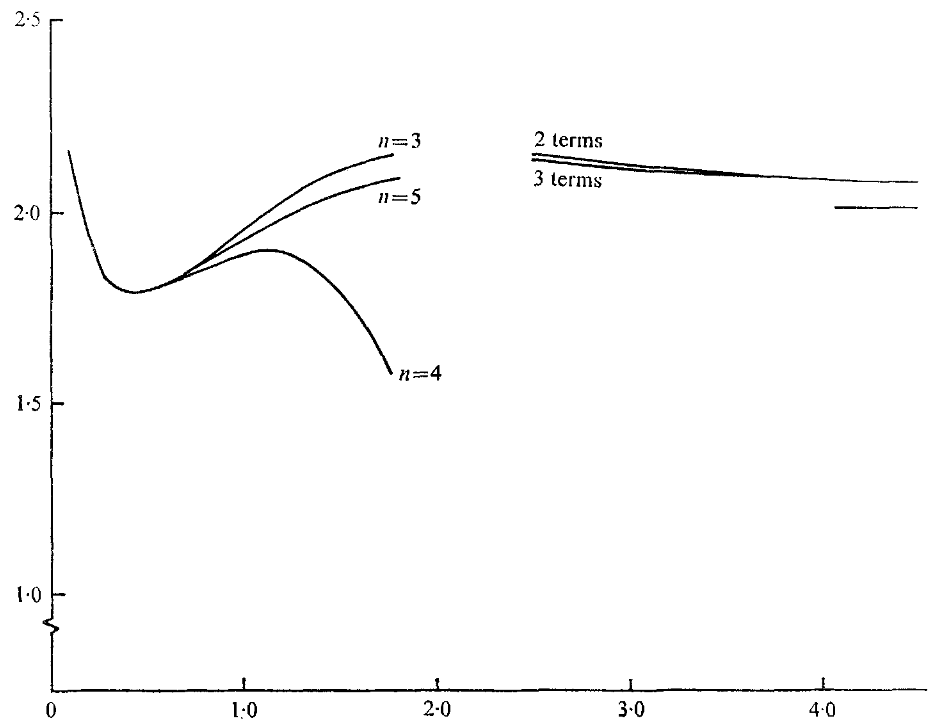

FIGURE 2. Variation of the second-order correction to skin friction with x for the cascade model. The curves for small x are obtained from the series (3.8) summed to 3, 4 and 5 terms. Also shown is the asymptotic solution for large x. This is obtained from (3.23) and (3.24) and the curves show 2 and 3 terms. The asymptote $Rc_f = 2{\cdot}02$ is indicated.

A small circular region of size R^{-1} surrounding the leading edge has been ignored. It seems likely that this will not affect the expansions until $O(R^{-1})$ is reached (at worst). (The effect of the corner is that the boundary condition of (3.2) must be abandoned within a distance $O(R^{-1})$ of $x = 0$, but this will have only local effects because the equation is elliptic.) At any rate it is reasonable to pursue the boundary-layer expansion, disregarding eigensolutions wherever possible, because the expansions in the various regions must be mutually consistent.

Finally, we note that the analysis of § 3 (i) may be used to throw some light on a somewhat different problem. If the flow of a stream over a semi-infinite flat plate is slightly disturbed, so that $U(x) = U_0 + U_1(x)$ for $x > 0$ where $|U_1| \ll U_0$ and

$U_1 \to 0$ as $x \to \infty$ then the asymptotic Blasius solution for large x has the effective origin at $x = -l$, where

$$l = 2kU_0^{-1} \int_0^\infty U_1(x)\,dx. \tag{4.1}$$

This result offers a parallel to Stewartson's (1957) equation for the leading-edge shift in the case where $U = U_0$ and the velocity profile is given at $x = 0$, though (4.1) requires that the disturbance to the Blasius solution should be small.

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